F. Gerbier, P. Bouyer and A. Aspect

Groupe d'Optique Atomique, Laboratoire Charles Fabry de l'Institut d'Optique, UMRA 8501 du CNRS, Bât. 503, Campus universitaire d'Orsay, B.P. 147, F-91403 ORSAY CEDEX, FRANCE

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We analyse the extraction of a coherent atomic beam from a trapped Bose-Einstein condensate using a rf transition to a non-trapping state at T=0 K. Our quantum treatment fully takes gravity into account but neglects all interactions in the free falling beam. We obtain an analytical expression of the output rate and of the wave function of the extracted beam, *i.e.* the output mode of the "atom laser". Our model reproduces satisfactorily experimental data without any adjustable parameter.

Bose-Einstein condensates (BEC) of dilute alkali vapors [1] represents a potential source of matter waves for atom interferometry, since it has been proven [2] that they are inherently coherent. Various schemes for "atom lasers" have been used to extract a coherent matter wave out of a trapped BEC. Pulsed devices were demonstrated by using a spin-flip radio frequency (rf) pulse [3], Raman transitions [4] or gravity-induced tunneling from an optically trapped BEC [5]. Later on, a quasi-continuous atom laser has been demonstrated by using a weak rf field that continuously couples atoms into a free falling state [6]. This "quasi-continuous" atom laser promises spectacular improvements in applications of atom optics, for example in the performances of atom-interferometer-based inertial sensors [7].

Gravity plays a crucial role in outcouplers with spin-flip rf transitions: it determines the direction of propagation of the matter wave, as well as its amplitude and phase. However, to our knowledge, most of the theoretical studies of rf couplers do not take gravity into account. This is the case in the numerical treatments of [8–10], and in the semi-classical analytical approaches of [11,12]. Gravity has been included in a 3D numerical treatment restricted to very short pulses [14], relevant for the experiment of [3], and in the 1D simulation of [13], relevant for the experiment of [6], with which it agrees only qualitatively.

In this paper, we present a quantum 3D analytical treatment fully taking gravity into account. After a short presentation of the basic equations, assumptions and approximations of our model, we identify the weak coupling regime relevant to the quasi-continuous atom laser. We obtain an analytic expression for the atom laser wave function and for the output rate, from which we derive a generalized rate equation for the trapped BEC. We then show that our model reproduces satisfactorily the experimental results of [6], without any adjustable parameter.

We consider a ^{87}Rb BEC in the $F = 1$ hyperfine level at $T = 0\ K$. The m =-1 state is confined in a harmonic magnetic potential $V_{\rm trap} = \frac{1}{2}M(\omega_x^2 x^2 + \omega_\perp^2 y^2 + \omega_\perp^2 z^2)$. A rf magnetic field $\mathbf{B}_{\rm rf} = B_{\rm rf}\cos(\omega_{\rm rf}t)\mathbf{e_x}$ can induce transitions to m =0 (non-trapping state) and m =+1 (expelling state). The coupling matrix element is $\hbar\Omega_{\rm rf}/2$, with the Rabi frequency $\Omega_{\rm rf} = \mu_B B_{\rm rf}/\hbar 2\sqrt{2}$ (g_F=-1/2 is the Landé g factor). Within the Hartree-Fock approximation, the condensate three-component spinor wavefunction $\mathbf{\Psi}' = [\psi_m']_{m=-1,0,+1}$ obeys a coupled non linear Schrödinger equation [8]. As in [11], we make the rotating wave approximation and the transformation $\psi_m{}' = \psi_m e^{im\omega_{\rm rf}t}$, and we consider the "weak coupling limit". In this limit to be defined more precisely later, the Rabi frequency is low enough that the populations N_m of the three Zeeman sublevels obey the following inequality: $N_{+1} \ll N_0 \ll N_{-1}$. In the rest of this paper, we therefore restrict ourselves to m =-1 and m =0, and set the total atomic density $n(\mathbf{r}) \approx\mid \psi_{-1}(\mathbf{r},t)\mid^2$. Finally, we neglect all interactions in the free falling state (we discuss later this approximation).

At this stage, we are left with the following two coupled equations:

$$i\hbar\frac{\partial\psi_{-1}}{\partial t} = h_{-1}\psi_{-1} + \frac{\hbar\Omega_{\rm rf}}{2}\psi_0 \tag{1a}$$

$$i\hbar\frac{\partial\psi_0}{\partial t} = h_0\psi_0 + \frac{\hbar\Omega_{\rm rf}}{2}\psi_{-1} \tag{1b}$$

with $h_{-1} = V_{\rm off} - \hbar\omega_{\rm rf} + \mathbf{p}^2/2M + V_{\rm trap} + U \mid \psi_{-1}\mid^2$ and $h_0 = \mathbf{p}^2/2M - Mgz$. The origin of the z axis is at the center of the condensate, displaced by gravity from the magnetic field minimum ($z_{\rm sag} = g/\omega_\perp^2$). We have taken the zero of energy at $z = 0$ in m =0, so that the level splitting at the bottom of the trap is $V_{\rm off} = \mu_B B_0/2 + Mg^2/2\omega_\perp^2$ (B_0 is the bias field). Interactions are described by $U = 4\pi\hbar^2 a/M$ ($a \approx 5$ nm).

We will expand the two components of the wavefunction onto the eigenstates ϕ_m of the uncoupled "effective hamiltonian" (taking $\Omega_{\rm rf} = 0$ in Eq.(1)). For the m =-1 sublevel, we solve for the ground state in the Thomas-Fermi (TF) approximation [15]

$$\phi_{-1}(\mathbf{r}) = \max\left[(\mu/U)^{1/2}[1-\tilde{r}_T^2-(\tilde{z})^2]^{1/2}\,,\,0\right] \qquad (2)$$

In the rest of the paper, we will use the reduced coordinates, $\tilde{r}_T^2{=}(x/x_0)^2+(y/y_0)^2$ in the transverse plane and $\tilde{z}{=}z/z_0$ in the vertical direction such that $\tilde{r}_T^2+\tilde{z}^2)\leq 1$. The BEC dimensions are respectively $x_0{=}(2\mu/M\omega_x^2)^{1/2}$ and $y_0{=}z_0{=}(2\mu/M\omega_{\perp^2}^2)^{1/2}$, while the chemical potential is $\mu{=}(15/8\pi)Ux_0y_0z_0$ ($\sigma{=}(\hbar/m\omega)^{1/2}$ and $\omega{=}(\omega_x\omega_\perp^2)^{1/3}$). The energy of the uncoupled eigenstates is $E_{-1}{=}V_{\rm off}+\mu-\hbar\omega_{rf}$. In the following, we will take typical values corresponding to the situation of [6]: $N=7\times10^5$ atoms initially, $\omega_x{=}2\pi\times 20$ Hz and $\omega_\perp{=}2\pi\times 180$ Hz, which gives $x_0\approx 55\ \mu$m , $z_0\approx 5\ \mu$m and $\mu/h\approx 2.2$ kHz.
For the m =0 state, the hamiltonian of equation (1b) with $\Omega_{\rm rf}$ =0 is separable. In the horizontal plane (x,y), the eigenstates are plane waves with wavevectors k_x,k_y that we quantize with periodic boundaries conditions in a 2D box of size L. Consquently, wavefunction is $\phi_0^\perp(x,y)\,=\,L^{-1}e^{i(k_xx+k_yy)}$ and the density of states is $\rho_{xy}\,=\,L^2/4\pi^2$. Along the vertical direction z, we must solve the usual 1D Schrödinger equation for a free falling particle [16]. The exact solution is $\phi_0^{(z_{E_z})}{=}\ A.Ai(-\zeta_{E_z})$, where Ai is the Airy function of the first kind taken for the reduced variable $\zeta_{E_z}\,=\,(z-z_{E_z})/l$. The classical turning point $z_{E_z}\,=\,-E_z/mg$ associated with the vertical energy E_z labels the vertical solution. We defined the length scale $l\,=\,(\hbar^2/2M^2g)^{1/3}$ such that $l\ll x_0,y_0,z_0$ (for ^{87}Rb, $l\approx 0.28\mu m$). In the following, we will rather use the WKB approximation which reads:

$$z\gtrsim z_{E_z}:\phi_0^{(z_{E_z})}=\frac{A}{\sqrt{\pi\mid\zeta_{E_z}\mid^{1/2}}}\cos(\frac{2}{3}\mid\zeta_{E_z}\mid^{3/2}-\frac{\pi}{4}) \qquad (3a)$$

$$z\lesssim z_{E_z}:\phi_0^{(z_{E_z})}=\frac{A}{2\sqrt{\pi\mid\zeta_{E_z}\mid^{1/2}}}e^{-\frac{2}{3}|\zeta_{E_z}|^{3/2}-i\frac{\pi}{4}} \qquad (3b)$$

This amounts to taking the asymptotic form of Ai. If we quantize $\phi_0^{(z_{E_z})}$ by imposing a node on a ''fictitious" barrier at position $z{=}H$ (that can be arbitrarily large), and neglect the contribution of the interval $[-\infty,z_{E_z}]$, where Ai falls off very quickly, we obtain the normalization factor $A{=}(\pi/l)^{1/2}\alpha_H^{-1}$, where $\alpha_H{=}((H-z_{E_z})/l)^{1/4}$ (we have replaced the rapidly oscillating $\cos^2$ term by 1/2). The density of longitudinal modes is $\rho_z(z_{E_z}){=}(1/2\pi l)\alpha_H^2$. Finally, the output modes are given by $\phi_0^{(n)}(\mathbf{r}){=}\phi_0^\perp(x,y)\phi_0^{(z_{E_z})}(z)$, where n stands for the quantum numbers (k_x,k_y,z_{E_z}). The density of modes is $\rho_{3D}{=}L^2\alpha_h^2/8\pi^3l$.

We thus have reduced the problem to the coupling of an initially populated bound state $\mathbf{\Psi_i}{=}\phi_{-1}\otimes\mid m=-1\rangle$, of energy $E_i{=}V_{\rm off}+\mu-\hbar\omega_{\rm rf}$ to a quasi-continuum of final states $\mathbf{\Psi_f}^{(n)}{=}\phi_0^{(n)}\otimes\mid m=0\rangle$, with a total energy $E_f^{(n)}{=}\hbar^2(k_x^2+k_y^2)/2M-Mgz_{E_z}$. A crucial feature in this problem is the resonant bell-shape of the coupling matrix element $W_{\rm fi}{=}(\hbar\Omega_{\rm rf}/2)\langle\phi_0^{(n)}\mid\phi_{-1}\rangle$. Because of the properties of the Airy function (see Appendix A and Ref. [17]), the above overlap integral is non-vanishing only if z_{E_z} belongs to $[-z_0,z_0]$ (within a negligible uncertainty of order l). This corresponds to accessible final energies $E_f^{(n)}\approx E_i$ restricted to an interval $\Delta\sim 2Mgz_0\sim 20kHz$, which gives a resonance condition for the frequency $\omega_{\rm rf}$ [6,10]:

$$\mid\hbar\omega_{\rm rf}-V_{\rm off}-\mu\mid\lesssim Mgz_0 \qquad (4)$$

Here and in the following, we neglect the transverse kinetic energy $\hbar^2k^2/2M$ (we will discuss this point later).

FIG. 1. Atom laser in the weak coupling regime. 1a : time evolution of the laser intensity. For $t\gg t_c$, the numerical integration of Eq.(1) agrees with the output rate of Eq.(5). 1b : Spatial intensity profile of the atom laser at $t\gg t_c$ according to Eq.(6)

We can describe the outcoupler dynamics by using the results of [18] for the coupling of a discrete level to a continuum. If the coupling matrix element takes significant values only for a finite range Δ in energy, two different situations can be distinguished. In the strong coupling regime ($h\Gamma\gg\Delta$, where Γ is the transition rate given by the Fermi golden rule), the discrete level exhibits Rabi oscillations with the narrow-band continuum. This describes the pulsed atom laser experimentally realized by Mewes *et al.* [3]). On the contrary, in the weak coupling limit ($h\Gamma\ll\Delta$), oscillations persist only for $t\leq t_c{=}h/\Delta\sim 0.05ms$. We have verified this behaviour on a 1D numerical integration analogous to [13] Fig.(1a). For $t\geq t_c$, the decay of the discrete level is monotonous with a rate Γ, which is calculated in Appendix A :

$$\frac{\Gamma}{\Omega_{\rm rf}^2}\approx\frac{15\pi}{32}\frac{\hbar}{\Delta}[1-(\frac{V_{\rm off}+\mu-\hbar\omega{\rm rf}}{\Delta/2})^2]^2 \qquad (5)$$

Quasi-continuous output corresponds to the weak coupling regime ($h\Gamma\ll\Delta$). From Eq.(5), we deduce a critical Rabi frequency (for which $h\Gamma\sim\Delta$) $\Omega_{\rm rf}^C\sim 0.8\Delta/\hbar$. To obtain Eq.(5), we have neglected the transverse kinetic energy. In this approximation, first order time dependent perturbation theory (valid for $t_c \leq t \leq \Gamma^{-1}$) also gives an analytical expression for the outcoupled atomic wave function, *i.e.* the atom laser mode (see Fig.(1b)):

$$\psi_0(\mathbf{r},t) \approx \begin{array}{c} -2A' i\pi \frac{\hbar\Omega_{rf}}{Mgl} e^{-i\frac{2}{3}|\zeta_{E_z}|^{3/2}} e^{-i\frac{\pi}{4}} e^{-i\frac{E_{-1}t}{2\hbar}} \\ \frac{1}{\sqrt{\pi|\zeta_{E_z}|^{1/2}}}\phi_{-1}(0,0,z_{E_i}) f^{\perp}(x,y) \end{array} \quad (6)$$

In this expression, we kept only the complex component of $\cos(\frac{2}{3} \mid \zeta_{E_z} \mid^{3/2} - \frac{\pi}{4})$ in Eq.(3a) that propagates downwards. The transverse profile is

$$f^{\perp}(x,y) = \int_0^{+\infty} dv (v\cos v - \sin v) J_0(v r_{\mathrm{out}})/v^2 \quad (7)$$

where J_0 is the Bessel function. We defined $r_{\mathrm{out}}^2 = (x/x_{\mathrm{out}})^2 + (y/y_{\mathrm{out}})^2$, with x_{out} (resp. y_{out}) $= x_0$ (resp. y_0) $(1-(2E_i/\Delta)^2)^{1/2}$. The constant A' can be calculated from the expression of Γ.

We now model the time evolution of the number of trapped atoms N_{-1}. Intuitively, one would write a rate equation using Eq.(5). However, the non linearity arising from the dependence of the output rate with N_{-1} should not be neglected and we assume that the condensed state adiabatically follows the evolution of the TF solution calculated with the time varying chemical potential [19]. With this approximation, the Wigner-Weisskopf treatment leads, after integration over space, to the non linear rate equation:

$$\frac{dN_{-1}}{dt} = -\Gamma[N_{-1}]\ N_{-1} \quad (8)$$

We can now compare our model to the data of Bloch *et al* [6]. Integrating Eq.(8) with the output rate (5), we have calculated the number of atoms remaining in the condensate after a fixed time with their experimental parameters as a function of the rf frequency. A shown Fig.(2a), the model agrees with the experimental data for the $\mid F=1; m_F=-1\rangle$. In the $\mid F=2; m_F=2\rangle$ case the second trapping state $\mid F=2; m_F=1\rangle$ has to be included. The resolvent operator allows to treat the resonant coupling between two discrete levels, with the first level also coupled to a continuum (decay rate $\Gamma_{2,1}$) [18]. If the coupling is strang ($\Gamma_{2,2} = \Gamma_{2,1}/2$ if $\Omega_{\mathrm{rf}} \gg \Gamma_{2,1}$), the second level acquires a decay rate $\Gamma_{2,2} = \Gamma_{2,1}/2$ if $\Omega_{\mathrm{rf}} \gg \Gamma_{2,1}$. We have taken this value as a first approximation. This leads to a good agreement with the experimental data (Fig.2b).

We now discuss our approximations in the case of weak coupling. Since $h\Gamma \ll mgz_0$, the spatial region where outcoupling takes place, of vertical extension $\delta z \sim h\Gamma/mg$, is very thin compared to the BEC size. This allows to loof at the outcoupling process in a semi classical way, in analogy with a Franck-Condon principle [9]: the coupling happens at the turning point of the classical trajectory of the free falling atoms. Using this semi classical point of view, we can verify our assumptions. Our first approximation consists of neglecting the transverse kinetic energy. A typical wavevector y_0^{-1} corresponds to a typical kinetic energy $\delta E \sim \hbar^2/2My_0^2 \ll \Delta$ ($\delta E \sim 20$ Hz for $y_0 \sim 5\mu m$). Neglecting δE amounts to approximate the Franck-Condon surfaces (surfaces of equal energy) by a plane. This is correct since, over the size of the BEC, the surface curvature is small, so that the deviation from this plane is negligible compared to z_0. This approximation also entails that the atom laser wave function has plane wave fronts [21]. Our second approximation consists of omitting the interaction term in the output. In other words, we have assumed that the gravitational potential acting on the free falling atoms was only slightly distorted by the interaction energy within the BEC. The validity of this can be checked by comparing the mean-field energy and the gradient of the gravitational potential over the condensate. This ratio is proportional to $\mu/\Delta \sim 0.1 \ll 1$. Gravity is thus the dominant term, although interactions may still play a significant role, especially in the transverse shape of the wave function.

To conclude, in this paper we have obtained analytical expressions for the output rate and the output mode of a quasi-continuous atom laser based on rf outcoupling from a trapped BEC. Our quantum treatment, which fully takes gravity and the 3D geometry into account, leads to a good agreement with the experimental results of Ref. [6]. Our model can easily be adapted to more sophisticated situations. For instance, it can describe the weak coupling of a trapped BEC to a magnetic cavity, *i.e.* a smooth magnetic trap for $m=0$ produced by the quadratic Zeeman effect [23]. Our model can still be improved by including the effects of interactions in the free beam, diffraction and finite-temperature [10].

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FIG. 2. Number of trapped atoms after 20 ms of rf outcoupling, starting with $\approx 7.2 \times 10^5$ condensed atoms, Ω_{rf}=312 Hz for the $\mid 1; -1\rangle$ sublevel (left), and $\approx 7.0 \times 10^5$ atoms, Ω_{rf}=700 Hz for the $\mid 2; 2\rangle$ sublevel (right). Circles are the experimental points from Bloch *et al.*, solid line is the prediction based upon our model using the experimental parameters. Theoretical and experimental curves have been shifted in frequency to match each other, since V_{off} is not experimentally known precisely enough (a precision of 10^{-3} Gauss for the bias field B_0 is required to know V_{off} within a kHz uncertainty).

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[19] This requires: $\mid (\partial h_{-1}/\partial t) \mid = \Gamma U \mid \psi_{-1} \mid^2 \ll \hbar\omega_{k0}^2$, so that no transitions from the ground state to the jth excited level (with frequency ω_{j0}) occur (see also the seconf of Ref. [10]).

[20] Results of a similar treatment with g=0 and plane waves as final states will be published elsewhere.

[21] Note that because of diffraction, this property will not persist beyond a distance z_R, equivalent to the Rayleigh length in photonic laser beams. It can be estimated by writing that the transverse spreading $(\hbar/My_{\mathrm{out}})(2z_R/g)^{1/2}$ becomes of the order of the size y_{out}, i.e. $z_R \approx y_{out}^4/4l^3$.

[22] Results of a similar treatment with g=0 and plane waves as final states will be published elsewhere.

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APPENDIX A: OUTPUT RATE

We detail here the evaluation of the output rate. We set $\tilde{k}^2 = (k_x x_0)^2 + (k_y y_0)^2$ and work out the overlap integral I between a free mode $\phi_0^{(n)}$ and the condensate wavefunction function ϕ_{-1} treated in the TF approximation. We write it in reduced cylindrical coordinates $(\tilde{r_T}, \theta, \tilde{z})$. Integration of $e^{i\tilde{k}\tilde{r_T}\cos\theta}$ over θ yields a $2\pi J_0$ term in the transverse integral, that we integrate over $\tilde{r_T}$ to obtain:

$$I = 2\pi \frac{A}{L}\sqrt{\frac{\mu}{U}} x_0 y_0 \int_{-z_0}^{z_0} dz Ai(-\frac{z - z_{E_z}}{l}) g[p(\tilde{k},\tilde{z})] \quad \text{(A1)}$$

where $g(p) = (p\cos p - \sin p)/p^3$, and $p(\tilde{k},\tilde{z}) = \tilde{k}(1 - \tilde{z}^2)^{1/2}$. We set $u = z/l$, and extend the integral over the real axis since $u_0 = z_0/l \gg 1$. Next, we use the Parseval relation, and introduce the Fourier transforms of g, $\tilde{g}$ and of $Ai(x)$, $e^{iu^3/3}$. We have to calculate $\int_{-\infty}^{\infty} \tilde{g}(\tilde{k}, vu_0) e^{i(v^3/3 - vu_{E_z})} dv$. As the function $\tilde{g}$ takes finite values on a small neighborhood of the origin (of size $u_0^{-1} \ll 1$), we can keep only the lowest order term in the phase, which gives the inverse Fourier transform of $\tilde{g}$ taken in $z = z_{E_z}$. This can be shown more rigorously with help of the Lebesgue theorem on uniform convergence [17]. This yields the simple expression:

$$I = 2\pi \frac{Al}{L}\sqrt{\frac{\mu}{U}} x_0 y_0 g(\tilde{k}, z_{\tilde{E}_z}) \quad \text{(A2)}$$

This expression together with the density of states permits to use the Fermi golden rule:

$$\Gamma = \frac{\pi^2}{2}\frac{\hbar\Omega_{\mathrm{rf}}^2}{Mg}\frac{\mu}{U} x_0 y_0 \int_0^{+\infty} \tilde{k} \mid g[p(\tilde{k}, \frac{E_{-1} - \frac{\hbar^2 k^2}{2M}}{Mgz_0})] \mid^2 d\tilde{k} \quad \text{(A3)}$$

Next we neglect the kinetic energy term before the potential energy: $\hbar^2 k^2/2M \ll E_{-1}$. The change of variable to $w = p(\tilde{k}, E_{-1}/Mg)$ yields a factor $(1-(E_{-1}/Mgz_0)^2)^2$ and $\int_0^{+\infty} dw \mid w\cos(w) - \sin(w) \mid^2 w^{-5}$ that we numerically found to be equal to $1/4$. Inserting these results into (A3),and using $8\pi\mu x_0 y_0 z_0 = 15U$ (in the TF limit), we obtain Eq. (5).